Killing Vectors in Spacetime of the De Sitter Invariant Special Relativity

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Abstract

In this paper, we use the Killing vector method to formulate the de Sitter/Anti-de Sitter invariant special relativity (dS/AdS-SR). Through solving the Einstein equation with $\Lambda \neq 0$, the basic inertial metric for dS/AdS-SR is determined to be the Beltrami metric $B_{\mu\nu}(x)$. The corresponding Killing equations are system of ten simultaneous partial differential equations of first order. Their most general solutions were obtained, and all the ten independent Killing vectors were found out. These results confirm that the Beltrami metric has maximal spacetime symmetry. The ten Killing-Noether charges are obtained. They are energy, momenta, Lorentz boost and angular momentum in SR-theory with $\Lambda \neq 0$. Consequently, dS/AdS-SR is consistently established for the vacuum with $\Lambda \neq 0$ via Killing vector method rather than the unpopular classical domain theory.

1 Introduction

Common Special Relativity (SR) is invariant under Poincaré transformations and its basic space-time metric is Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu} \equiv \mathrm{diag}\{+,-,-,-\}$, which satisfies the vacuum (or empty spacetime) Einstein equation without universal Einstein Cosmologic Constant (ECC) Λ. It is easy to see when $\Lambda \neq 0$, the Minkowski metric will no longer be a solution of the vacuum Einstein equation because a new term $\Lambda g_{\mu\nu}$ will emerge in the equation. In this case the common SR should naturally become the de Sitter (or Anti de Sitter) invariant Special Relativity (dS/AdS-SR) [1, 2](see also [3] and references within). In other words, the de Sitter/Anti de Sitter invariant Special Relativity is the Special Relativity in the vacuum spacetime with the non-zero universal Einstein Cosmology Constant Λ. It is essential that the vacuum of dS/AdS-SR is different from one of common SR.

About the end of last century, the accelerating expansions of the Universe were discovered [4, 5]. The accelerating expansions of the Universe indicate that there is an

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effective positive cosmologic constant $\Lambda_{\text{eff}} \equiv \Lambda + 8\pi G\rho_{\text{dark energy}} \neq 0$ in the real world spacetime, where Λ is ECC that is a universal constant, G is Newton gravitational constant and $\rho_{\text{dark energy}}$ is density of dark energies (e.g., see [6–8]). We address that the fact of $\Lambda_{eff} \neq 0$ in general means that ECC $\Lambda \neq 0$ and $\rho_{\text{dark energy}} \neq 0$. It is *ad hoc* to assume $\Lambda = 0$ artificially in the studies of the cosmology of the time. Thus, after this discover, the theory of dS/AdS-SR attracts great interests [3, 8–22].

The SR basic spacetime metric $g_{\mu\nu}$ plays a pivotal role in SR-theory, which can be determined by following requirements:

1. The $g_{\mu\nu}$ satisfies the vacuum Einstein equation.
2. In the $g_{\mu\nu}$ spacetime, the motions of free particles are inertial.
3. $g_{\mu\nu}$ has maximal spacetime symmetry.

In addition, it should be also kept in mind that the non-relativistic limit of any relativistic mechanics has to be the common Newtonian mechanics [23]. When ECC $\Lambda = 0$, the solution satisfying the above three requirements is $g_{\mu\nu} = \eta_{\mu\nu}$. When $\Lambda \neq 0$, the situation becomes non-trivial and challenging. For this case, based on analysis of symmetrical space by using classical domain theory method [24], a remarkable metric $g_{\mu\nu}(\lambda, x)$ was suggested in Ref. [2] (where λ is a constant), which satisfies the 2^{nd} requirement (i.e., inertial motion law for free particles holed in the spacetime with this $g_{\mu\nu}(\lambda, x)$). In this present paper, we take this classical domain spacetime metric $g_{\mu\nu}(\lambda, x)$ to be an ansatz for solving the vacuum Einstein equation with $\Lambda \neq 0$. We will find that $g_{\mu\nu}(\lambda, x)$ with $\lambda = \Lambda/3$ (see Eq. (8) below) is the solution, i.e., $g_{\mu\nu}(\lambda, x)|_{\lambda=\Lambda/3} \equiv B_{\mu\nu}(x)$ is the solution of the vacuum Einstein equation with $\Lambda \neq 0$, and it will be called Beltrami metric. Hence, $B_{\mu\nu}(x)$ satisfies both the 1^{st} requirement and 2^{nd} requirement. Next we should examine whether $B_{\mu\nu}(x)$ satisfies the 3^{rd} requirement. This is the main aim of the present paper. We will present explicit calculations to solve the Killing vector equation of $B_{\mu\nu}(x)$, which is a system of ten simultaneous partial differential equations of first order (e.g., see the 13th chapter in [25]), and all the corresponding Killing-Neother charges are found out.

The rest of the paper is organized as follows. In section II, we solve the vacuum Einstein equation with $\Lambda \neq 0$. In this way, we find that the basic metric of dS/AdS-SR is the Beltrami metric. In section III, we solve the Killing equation of Beltrami metric. It is a system of ten simultaneous partial differential equations of first order. All Killing vectors in the Beltrami spacetime are found out. Section IV devotes to calculating Killing-Noether charges, and confirms the metric $B_{\mu\nu}(x)$ has maximal spacetime symmetry. Finally, we briefly summarize and discuss our results in this paper.

2 Basic metric of dS/AdS-invariant special relativity

The Einstein equation with cosmologic constant is given by,

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu} = 0, \tag{1}$$

where $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ are the Ricci tensor and curvature scalar of 4-dimensional Riemann geometry respectively. In [2], the following metric ansatz was suggested,

$$g_{\mu\nu}(\lambda, x) = \frac{\eta_{\mu\nu}}{\sigma(\lambda, x)} + \frac{\lambda\eta_{\mu\alpha}\eta_{\nu\beta}x^\alpha x^\beta}{\sigma(\lambda, x)^2}, \quad \text{with} \quad \sigma(\lambda, x) = 1 - \lambda\eta_{\mu\nu}x^\mu x^\nu\,, \tag{2}$$

where λ is a constant. Straightforward geometry calculations give us

$$g^{\mu\nu}(\lambda,x)=\sigma(\lambda,x)(\eta^{\mu\nu}-\lambda x^\mu x^\nu), \tag{3}$$

$$\Gamma^\rho_{\mu\nu}(\lambda,x)=\frac{\lambda}{\sigma(\lambda,x)}(\delta^\rho_\mu\eta_{\nu\lambda}x^\lambda+\delta^\rho_\nu\eta_{\mu\lambda}x^\lambda), \tag{4}$$

$$\mathcal{R}^\rho{}_{\lambda\mu\nu}(\lambda,x)=\lambda\left[g_{\lambda\mu}(\lambda,x)\delta^\rho_\nu-g_{\lambda\nu}(\lambda,x)\delta^\rho_\mu\right], \tag{5}$$

$$\mathcal{R}_{\mu\nu}(\lambda,x)=3\lambda g_{\mu\nu}(\lambda,x), \tag{6}$$

$$\mathcal{R}(\lambda)=12\lambda=\text{constant}. \tag{7}$$

Substituting Eqs. (6, 7) into Eq. (1), we obtain

$$\lambda=\frac{\Lambda}{3}. \tag{8}$$

Consequently, the solution of the vacuum Einstein equation is

$$g_{\mu\nu}(x)\equiv B_{\mu\nu}(x)=\frac{\eta_{\mu\nu}}{\sigma(x)}+\frac{\Lambda\eta_{\mu\alpha}\eta_{\nu\beta}x^\alpha x^\beta}{3\sigma(x)^2},\quad \text{with}\ \ \sigma(x)\equiv\sigma(\lambda,x)|_{\lambda=\Lambda/3}=1-\frac{\Lambda}{3}\eta_{\mu\nu}x^\mu x^\nu. \tag{9}$$

We call $B_{\mu\nu}(x)$ the Beltrami metric, and hence the 1^{st} requirement listed in the last section is satisfied. The metric $g_{\mu\nu}(x)$ which is the solution to Eq. (1) have dS/AdS-spacetime symmetry [1].

The inertial motion law for free particle in the Beltrami spacetime $\mathcal{B}$ described by $B_{\mu\nu}(x)$ has been discussed in [2, 3, 16]. In order to clarify the notations which will be used below, we shall recapitulate the key points here. The inertial motion law in $\mathcal{B}$ requires that the free particles in $\mathcal{B}$ move uniformly along the straight line (or geodesic). Namely, by means of the principle of least action (which is the equivalent of the equation of motion along geodesic line in $\mathcal{B}$):

$$\delta S\equiv\delta\left[-mc\int ds\right]=-mc\,\delta\int\sqrt{B_{\mu\nu}(x)dx^\mu dx^\nu}=0\,, \tag{10}$$

we can get the solution as follows,

$$\ddot{\boldsymbol{x}}=0,\quad\text{or}\quad \boldsymbol{v}=\dot{\boldsymbol{x}}=\text{constant}, \tag{11}$$

where $S=-mc\int ds$ is Landau-Lifshitz action for free particle [23]and $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ are the velocity and the acceleration respectively. The solution of Eq. (11) have been obtained by solving Eq. (10) in Refs. [3, 16], and hence the 2^{nd} requirement listed in the last section is satisfied in the Beltrami spacetime $\mathcal{B}$. From the Landau-Lifshitz action $S=\int Ldt$ in $\mathcal{B}$-spacetime, we have

$$L=-mc\frac{ds}{dt}=-mc\frac{\sqrt{B_{\mu\nu}(x)dx^\mu dx^\nu}}{dt}=-mc\sqrt{B_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}. \tag{12}$$

Substituting (9) into (12) gives

$$L=-mc^2\sqrt{\frac{9(c^2-\dot{\boldsymbol{x}}^2)+3\Lambda[-\boldsymbol{x}^2\dot{\boldsymbol{x}}^2+(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})^2+c^2(\boldsymbol{x}-\dot{\boldsymbol{x}}t)^2]}{c^2[3+\Lambda(\boldsymbol{x}^2-c^2t^2)]^2}}. \tag{13}$$

It is easy to see when $\Lambda\to 0$we have

$$L\longrightarrow L_{Eins}=-mc^2\sqrt{1-\frac{\dot{\boldsymbol{x}}^2}{c^2}}, \tag{14}$$

where L_{Eins} is well known Lagrangian of common SR which is Pioncaré invariant [23]. By using the Euler-Lagrangian equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} - \frac{\partial L}{\partial \boldsymbol{x}} = 0, \tag{15}$$

and noting $L = L(t, \boldsymbol{x}, \dot{\boldsymbol{x}})$, $\partial/\partial\boldsymbol{x} \equiv \nabla = (\partial/\partial x^1)\boldsymbol{i} + (\partial/\partial x^2)\boldsymbol{j} + (\partial/\partial x^3)\boldsymbol{k}$, we can also obtain Eq. (11). The calculations are straightforward and non-trivial [3, 16].

In following section we will focus on the 3^{rd} requirement for basic metric of SR. We shall solve the Kiling vector equation to examine whether $B_{\mu\nu}(x)$ has maximal symmetry or not.

3 Killing vectors in Beltrami spacetime

In order to understanding the geometry of $\mathcal{B}$, and further to reveal the conservation laws in the mechanics of dS/AdS-SR, we derive the Killing vectors in this section. The metric in the Betrami spacetime $\mathcal{B}$ is $g_{\mu\nu}(x) = B_{\mu\nu}(x)$. Considering a infinitely small coordinate transformation:

$$x^\mu \longrightarrow x'^\mu = x^\mu + \epsilon\xi^\mu(x), \quad \text{with} \quad |\epsilon| << 1\,, \tag{16}$$

where $\xi^\mu(x)$ is generators of the transformation, the condition that $g_{\mu\nu}(x)$ is invariant under this transformation is given by

$$\mathcal{L}_\xi g_{\mu\nu}(x) = 0, \tag{17}$$

where $\mathcal{L}_\xi g_{\mu\nu}(x)$ is the Lee derivative of $g_{\mu\nu}(x)$, and then $\xi^\mu(x)$ is the Killing vector. Hence $\xi^\mu(x)$ is determined by following Killing vector equation (see, e.g., the 13th chapter of [25]):

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0\,, \tag{18}$$

where the covariant derivative $\xi_{\mu;\nu} = \xi_{\mu,\nu} - \Gamma^\lambda_{\mu\nu}\xi_\lambda$. For all possible values of μ and ν, the Killing equation of Eq. (18) reads as

$$\frac{\partial\xi_0}{\partial x^0} = \frac{2\Lambda x^0}{3\sigma(x)}\xi_0, \tag{19}$$

$$\frac{\partial\xi_1}{\partial x^1} = \frac{-2\Lambda x^1}{3\sigma(x)}\xi_1, \tag{20}$$

$$\frac{\partial\xi_2}{\partial x^2} = \frac{-2\Lambda x^2}{3\sigma(x)}\xi_2, \tag{21}$$

$$\frac{\partial\xi_3}{\partial x^3} = \frac{-2\Lambda x^3}{3\sigma(x)}\xi_3, \tag{22}$$

$$\frac{\partial\xi_0}{\partial x^1} + \frac{\partial\xi_1}{\partial x^0} = \frac{2\Lambda}{3\sigma(x)}(-\xi_0 x^1 + \xi_1 x^0)\,, \tag{23}$$

$$\frac{\partial\xi_0}{\partial x^2} + \frac{\partial\xi_2}{\partial x^0} = \frac{2\Lambda}{3\sigma(x)}(-\xi_0 x^2 + \xi_2 x^0)\,, \tag{24}$$

$$\frac{\partial\xi_0}{\partial x^3} + \frac{\partial\xi_3}{\partial x^0} = \frac{2\Lambda}{3\sigma(x)}(-\xi_0 x^3 + \xi_3 x^0)\,, \tag{25}$$

$$\frac{\partial\xi_1}{\partial x^2} + \frac{\partial\xi_2}{\partial x^1} = \frac{-2\Lambda}{3\sigma(x)}(\xi_2 x^1 + \xi_1 x^2)\,, \tag{26}$$

$$\frac{\partial\xi_1}{\partial x^3} + \frac{\partial\xi_3}{\partial x^1} = \frac{-2\Lambda}{3\sigma(x)}(\xi_3 x^1 + \xi_1 x^3)\,, \tag{27}$$

$$\frac{\partial\xi_2}{\partial x^3} + \frac{\partial\xi_3}{\partial x^2} = \frac{-2\Lambda}{3\sigma(x)}(\xi_2 x^3 + \xi_3 x^2)\,, \tag{28}$$

where $\sigma(x) = 1 - \frac{\Lambda}{3}\eta_{\mu\nu}x^\mu x^\nu$. Our purpose is to solve the above ten simultaneous partial differential equations of first order. For convenience and notation compactness, we introduce the notations $f(x^1, x^2, x^3) \equiv f(\not{0})$, $f(x^0, x^2, x^3) \equiv f(\not{1})$, $f(x^0, x^1, x^3) \equiv f(\not{2})$ and $f(x^0, x^1, x^2) \equiv f(\not{3})$. That is to say, $f(\not{\mu})$ is a multivariable function of x, but is independent of the μ-th component x^μ. From Eq. (19), we can obtain

$$\frac{d\xi_0}{\xi_0} = \frac{2\Lambda x^0 dx^0}{3(1 - \frac{\Lambda}{3}\eta_{\mu\nu}x^\mu x^\nu)} = \frac{\Lambda d[(x^0)^2]}{3 - \Lambda((x^0)^2 - \boldsymbol{x}^2)} \, .$$

Performing integrals on both sides of the above equation, we have

$$\ln \xi_0 = \ln \left[\frac{c(\not{0})}{\sigma(x)}\right] \Rightarrow \xi_0 = \frac{c(\not{0})}{\sigma(x)} \, . \tag{29}$$

In a similar way, from the equations (20), (21), (22), the following relations can be obtained,

$$\xi_1 = \frac{c(\not{1})}{\sigma(x)}, \quad \xi_2 = \frac{c(\not{2})}{\sigma(x)}, \quad \xi_3 = \frac{c(\not{3})}{\sigma(x)}, \tag{30}$$

Substituting Eqs. (29, 30) into (23)–(28), we have

$$\frac{\partial c(\not{1})}{\partial x^2} + \frac{\partial c(\not{2})}{\partial x^1} = 0 \, , \tag{31}$$
$$\frac{\partial c(\not{1})}{\partial x^3} + \frac{\partial c(\not{3})}{\partial x^1} = 0 \, , \tag{32}$$
$$\frac{\partial c(\not{1})}{\partial x^0} + \frac{\partial c(\not{0})}{\partial x^1} = 0 \, , \tag{33}$$
$$\frac{\partial c(\not{2})}{\partial x^3} + \frac{\partial c(\not{3})}{\partial x^2} = 0 \, , \tag{34}$$
$$\frac{\partial c(\not{2})}{\partial x^0} + \frac{\partial c(\not{0})}{\partial x^2} = 0 \, , \tag{35}$$
$$\frac{\partial c(\not{3})}{\partial x^0} + \frac{\partial c(\not{0})}{\partial x^3} = 0 \, . \tag{36}$$

As a consequence, we see that $\frac{\partial c(\not{i})}{\partial x^j}$ is independent of both x^i and x^j, and

$$\frac{\partial^3 c(\not{0})}{\partial x^1 \partial x^2 \partial x^3}, \quad \frac{\partial^3 c(\not{1})}{\partial x^0 \partial x^2 \partial x^3}, \quad \frac{\partial^3 c(\not{2})}{\partial x^0 \partial x^1 \partial x^3}, \quad \frac{\partial^3 c(\not{3})}{\partial x^0 \partial x^1 \partial x^2} \tag{37}$$

are constants. Hence the most general form of the function $c(\not{\mu})$ is as follows

$$\begin{aligned}
c(\not{0}) &= a_0 + b_{01}x^1 + b_{02}x^2 + b_{03}x^3 + d_{03}x^1x^2 + d_{02}x^1x^3 + d_{01}x^2x^3 + f_0 x^1x^2x^3 \, , \\
c(\not{1}) &= a_1 + b_{10}x^0 + b_{12}x^2 + b_{13}x^3 + d_{13}x^0x^2 + d_{12}x^0x^3 + d_{10}x^2x^3 + f_1 x^0x^2x^3 \, , \\
c(\not{2}) &= a_2 + b_{20}x^0 + b_{21}x^1 + b_{23}x^3 + d_{23}x^0x^1 + d_{21}x^0x^3 + d_{20}x^1x^3 + f_2 x^0x^1x^3 \, , \\
c(\not{3}) &= a_3 + b_{30}x^0 + b_{31}x^1 + b_{32}x^2 + d_{32}x^0x^1 + d_{31}x^0x^2 + d_{30}x^1x^2 + f_3 x^0x^1x^2 \, ,
\end{aligned} \tag{38}$$

where a_i, b_{ij}, d_{ij} and f_i with $i, j = 0, 1, 2, 3$ are real. Inserting Eq. (38) into Eqs. (31, 32, 33, 34, 35, 36), we obtain the following constraints

$$\begin{aligned}
&d_{01} = d_{02} = d_{03} = d_{10} = d_{12} = d_{13} = d_{20} = d_{21} = d_{23} = d_{30} = d_{31} = d_{32} = 0 \, , \\
&f_0 = f_1 = f_2 = f_3 = 0, \quad b_{01} = -b_{10}, \quad b_{02} = -b_{20}, \quad b_{03} = -b_{30} \, , \\
&b_{12} = -b_{21}, \quad b_{13} = -b_{31}, \quad b_{23} = -b_{32} \, .
\end{aligned} \tag{39}$$

Therefore the Killing vector of the Betrami metric is

$$\xi_\mu(x) = \frac{3c(\not{\mu})}{3 - \Lambda\eta_{\mu\nu}x^\mu x^\nu}, \tag{40}$$

with

$$\begin{pmatrix} c(\not{0}) \\ c(\not{1}) \\ c(\not{2}) \\ c(\not{3}) \end{pmatrix} = \begin{pmatrix} 0 & -b_{10} & -b_{20} & -b_{30} \\ b_{10} & 0 & b_{12} & b_{13} \\ b_{20} & -b_{12} & 0 & b_{23} \\ b_{30} & -b_{13} & -b_{23} & 0 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \tag{41}$$

where $b_{\mu\nu}$ and a_μ are ten independent constants, and hence Eq. (40) indicates that there are ten independent Killing vectors in the Beltrami spacetime $\mathcal{B}$ described by $g_{\mu\nu}(x) = B_{\mu\nu}(x)$. Noting the dimension of $\mathcal{B}$ is $N = 4$, and $N(N+1)/2 = 10$. Consequently the Beltrami metric $B_{\mu\nu}(x)$ has maximum spacetime symmetry (see, e.g., the 13th chapter of [25]), and then we have proved that the 3^{rd} requirement listed in the introduction section is satisfied for $B_{\mu\nu}(x)$. From (9), we can read off the contravariant metric tensor in $\mathcal{B}$ as

$$B^{\mu\nu}(x) = \sigma(x)\left(\eta^{\mu\nu} - \frac{\Lambda}{3}x^\mu x^\nu\right). \tag{42}$$

Consequently the contravariant Killing vector in $\mathcal{B}$ is

$$\xi^\mu(x) = B^{\mu\nu}(x)\xi_\nu(x) = \eta^{\mu\nu}c(\not{\nu}) - \frac{\Lambda}{3}x^\mu x^\nu c(\not{\nu}), \tag{43}$$

where $\eta^{\mu\nu}c(\not{\nu})$ refers to $\sum_{\nu=0}^{3}(\eta^{\mu\nu}c(\not{\nu}))$. Substituting (43) into (16) gives

$$x^\mu \longrightarrow x'^\mu = x^\mu + \epsilon\left(\eta^{\mu\nu}c(\not{\nu}) - \frac{\Lambda}{3}x^\mu x^\nu c(\not{\nu})\right), \quad \text{where} \quad |\epsilon| << 1, \tag{44}$$

which is the infinitely small coordinate transformation preserved by the Beltrami metric. Hence we can conclude that the Betrami metric (9) fully satisfies the three requirements for the basic spacetime metric of SR claimed in the introduction section.

4 Noether theorem and Killing-Noether charges

For clarifying the notations we briefly review the well known Noether theorem (see, e.g., [26, 27]) at first, and then we present detailed calculations for Killing-Noether charges in the following.

(A) Noether theorem

Considering a mechanics system, its dynamical behaviors are described by the Lagrangian $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and the Euler-Lagrange equation arising from the variation $\delta\int L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})dt = 0$. If the action $S \equiv \int L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})dt$ is invariant under the following space-time transformation

$$t \longrightarrow T, \quad \boldsymbol{q} \longrightarrow \boldsymbol{Q}. \tag{45}$$

In other words, we have

$$\int L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})dt = \int L(T, \boldsymbol{Q}, \acute{\boldsymbol{Q}})dT, \tag{46}$$

where $\acute{\boldsymbol{Q}} \equiv d\boldsymbol{Q}/dT$. Then, Noether theorem claims that the invariance of the action under (45) will lead to existence of certain motion constants which are called Noether

charges. When the transformations are generated by Killing vectors, the corresponding charges are called Killing-Noether charges.

Let's consider an infinitely small transformation, we write T and $\boldsymbol{Q}$ in Eq. (45) as follows,

$$T = T(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \epsilon), \tag{47}$$

$$\boldsymbol{Q} = \boldsymbol{Q}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \epsilon)\,, \tag{48}$$

where ϵ is an infinitesimal parameter being independent of the spacetime coordinates, and the following conditions hold,

$$T|_{\epsilon=0} = t, \tag{49}$$

$$\boldsymbol{Q}|_{\epsilon=0} = \boldsymbol{q}\,. \tag{50}$$

The function $\acute{\boldsymbol{Q}}$ in the right-handed side of Eq. (46) is then

$$\acute{\boldsymbol{Q}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \epsilon) \equiv \frac{d\boldsymbol{Q}}{dT} = \frac{d\boldsymbol{Q}/dt}{dT/dt} = \frac{\dot{\boldsymbol{Q}}}{\dot{T}} = \frac{\dot{\boldsymbol{Q}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \epsilon)}{\dot{T}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \epsilon)}\,. \tag{51}$$

We can also rewrite Eq. (46) as

$$\int [L(T, \boldsymbol{Q}, \acute{\boldsymbol{Q}})\dot{T} - L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})]dt = 0, \tag{52}$$

then it can be proved that the following parameter is a motion integral constant [3, 26, 27]:

$$G \equiv L\zeta + \sum_i \frac{\partial L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{q}^i}(\eta^i - \dot{q}^i\zeta) \tag{53}$$

where

$$\zeta = \left.\frac{\partial T(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}, \quad \eta^i = \left.\frac{\partial Q^i(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}. \tag{54}$$

Namely the conserved quantity G of Eq. (53) satisfies

$$\dot{G} = 0\,. \tag{55}$$

(B) Killing-Noether Charges

Based on Killing vector equations Eqs. (17, 18) it can be showed [25] that the infinitesimal transformation $x^\mu \to x'^\mu = x^\mu + \epsilon\xi^\mu$ leaves the Beltrami metric intact, i.e.

$$B_{\mu\nu}(x) \to B'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}B_{\alpha\beta}(x) = B_{\mu\nu}(x'). \tag{56}$$

Then it is easy to check that the Landau-Lifshitz action in Eq. (12) is invariant under this metric preserved transformation,

$$S \equiv -mc\int\sqrt{B_{\mu\nu}(x)dx^\mu dx^\nu} \to S' \equiv -mc\int\sqrt{B'_{\mu\nu}(x')dx'^\mu dx'^\nu} = -mc\int\sqrt{B_{\mu\nu}(x')dx'^\mu dx'^\nu} = S. \tag{57}$$

Therefore, using the expressions of ten independent Killing vectors in Eq. (43) and the Noether theorem of Eq. (53), the ten conserved quantities for dS/AdS-mechanics can be calculated out analytically.

1. Energy

Taking the constants in the Killing vector to be: $b_{\mu\nu}=0,\ a_1=a_2=a_3=0,\ a_0=-c$, noting $x^0=ct$, and substituting them into Eqs. (41, 43, 44), we obtain

$$t'=t+\frac{\epsilon}{c}\xi^0=t-\epsilon\left(1-\frac{\Lambda}{3}c^2t^2\right), \tag{58}$$

$$x'^i=x^i\left(1+\frac{\Lambda c^2t\epsilon}{3}\right). \tag{59}$$

Comparing Eq. (16) with Eq. (45) further, we have

$$t'=T, \qquad \boldsymbol{x}'=\boldsymbol{Q}. \tag{60}$$

Thus the parameters ζ and η^i defined in Eq. (54) take the form

$$\zeta=-1+\frac{\Lambda c^2t^2}{3}, \qquad \eta^i=x^i\frac{\Lambda c^2t}{3}. \tag{61}$$

The corresponding Noether charge denoted as G_{a^0} is given by

$$G_{a^0}=L\left(-1+\frac{\Lambda c^2t^2}{3}\right)+\sum_{i=1}^{3}\left[x^i\frac{\Lambda c^2t}{3}-\dot{x}^i\left(-1+\frac{\Lambda c^2t^2}{3}\right)\right]\left[\frac{m^2c^2}{L}\right.$$
$$\left.\times\frac{-9\dot{x}^i+3\Lambda[-\boldsymbol{x}^2\dot{x}^i+(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})x^i-c^2t(x^i-\dot{x}^it)]}{[3+\Lambda(\boldsymbol{x}^2-c^2t^2)]^2}\right]. \tag{62}$$

Inserting the expression of L in Eq. (13) into this equation, through an analytical calculation, we obtain

$$G_{a^0}\equiv E=\frac{mc^2}{\sqrt{1-\frac{\dot{\boldsymbol{x}}^2}{c^2}+\frac{\Lambda(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})^2-\Lambda\boldsymbol{x}^2\dot{\boldsymbol{x}}^2}{3c^2}+\frac{\Lambda(\boldsymbol{x}-\dot{\boldsymbol{x}}t)^2}{3}}}, \tag{63}$$

which is desired energy formula for dS/AdS-SR mechanics. Moreover, we would like to make two remarks as follows,

- Introducing dS/AdS-SR Lorentz factor

$$\Gamma\equiv\frac{1}{\sqrt{1-\frac{\dot{\boldsymbol{x}}^2}{c^2}+\Lambda\left[\frac{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})^2-\boldsymbol{x}^2\dot{\boldsymbol{x}}^2}{3c^2}+\frac{(\boldsymbol{x}-\dot{\boldsymbol{x}}t)^2}{3}\right]}}, \tag{64}$$

  then the energy in Eq. (63) and the Lagrangian in Eq. (13) can be compactly written as

$$E=mc^2\Gamma, \tag{65}$$

$$L=-mc^2(\sigma\Gamma)^{-1}, \tag{66}$$

  where σ is given in Eq. (9). In the limit of $\Lambda\to 0$, we have $\Gamma\to\gamma\equiv(1-\dot{\boldsymbol{x}}^2/c^2)^{-1/2}$, where γ is usual Lorentz contraction factor of common SR. The energy $E=mc^2\Gamma$ goes back to common SR's energy formula $E=mc^2\gamma$, L back to common Lagrangian of SR in Eq. (14). Therefore it is reasonable to identify the Noether-charge G_{a^0} as energy.

- From the equation of motion $\ddot{\boldsymbol{x}} = 0$ given in Eq. (11), it is easy to check that the equality $\dot{\Gamma} = 0$ is fulfilled. Consequently we have

$$\dot{E} = mc^2\dot{\Gamma} = 0\,. \tag{67}$$

The energy conservation is verified in the dS/AdS-SR mechanics. Noting even though the Lagrangian for dS/AdS-SR in Eq. (13) is time dependent, the corresponding energy is still conserving. This is a non-trivial character of the dS/AdS-SR Lagrangian formalism.

2. Momentum

Choosing the constants in the Killing vector to be $b_{\mu\nu} = 0,\ a_0 = a_2 = a_3 = 0,\ a_1 = -1$, accordingly the spacetime transformation is of the form

$$t' = t + \epsilon\frac{\Lambda}{3}tx^1, \tag{68}$$

$$x'^i = x^i + \epsilon\left(\delta^{i1} + \frac{\Lambda}{3}x^i x^1\right), \qquad i = 1,\ 2,\ 3\,, \tag{69}$$

which lead to

$$\zeta = \frac{\Lambda}{3}tx^1, \quad \eta^1 = 1 + \frac{\Lambda(x^1)^2}{3}, \quad \eta^2 = \frac{\Lambda x^1 x^2}{3}, \quad \eta^3 = \frac{\Lambda x^1 x^3}{3}\,. \tag{70}$$

Then we can straightforwardly determine the Noether charge is

$$G_{a^1} \equiv p^1 = \frac{m\dot{x}^1}{\sqrt{1 - \frac{\dot{\boldsymbol{x}}^2}{c^2} + \Lambda\left(\frac{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})^2 - \boldsymbol{x}^2\dot{\boldsymbol{x}}^2}{3c^2} + \frac{(\boldsymbol{x}-\dot{\boldsymbol{x}}t)^2}{3}\right)}}\,. \tag{71}$$

Similarly, setting $a_i = -1$ ($i = 2$ or 3), and other parameters $\{b_{\mu\nu},\ a_i\}$ in the Killing vector (43) vanish, the resulting conserved quantity is fixed to be

$$G_{a^i} \equiv p^i = \frac{m\dot{x}^i}{\sqrt{1 - \frac{\dot{\boldsymbol{x}}^2}{c^2} + \Lambda\left[\frac{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})^2 - \boldsymbol{x}^2\dot{\boldsymbol{x}}^2}{3c^2} + \frac{(\boldsymbol{x}-\dot{\boldsymbol{x}}t)^2}{3}\right]}} = m\dot{x}^i\Gamma\,. \tag{72}$$

Noting $\ddot{\boldsymbol{x}} = 0$ and $\dot{\Gamma} = 0$, the momentum conservation law also holds in dS/AdS-SR mechanics,

$$\dot{p}^i = 0, \quad \text{or} \quad \dot{\boldsymbol{p}} = 0\,. \tag{73}$$

3. Lorentz boost

In the same fashion as previous cases, taking constants in the Killing vector to be: $b_{10} = 1$ and other $b_{\mu\nu} = 0$, $a_0 = a_1 = a_2 = a_3 = 0$, we have

$$\begin{aligned} &t' = t - \frac{\epsilon x^1}{c}, \\ &x'^1 = x^1 - \epsilon ct, \quad x'^2 = x^2, \quad x'^3 = x^3, \end{aligned} \tag{74}$$

and

$$\zeta = \frac{-x^1}{c}, \quad \eta^1 = -ct, \quad \eta^2 = \eta^3 = 0. \tag{75}$$

The conserved quantity for this symmetry is given by

$$G_{b_{10}} \equiv K^1 = \frac{mc(x^1 - t\dot{x}^1)}{\sqrt{1 - \frac{\dot{\boldsymbol{x}}^2}{c^2} + \Lambda\left[\frac{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})^2 - \boldsymbol{x}^2\dot{\boldsymbol{x}}^2}{3c^2} + \frac{(\boldsymbol{x}-\dot{\boldsymbol{x}}t)^2}{3}\right]}} . \tag{76}$$

Similarly for the case of $b_{i0} = 1$ ($i = 2, 3$) and other parameters in the Kiling vector vanishing, we find the corresponding Lorentz boost Noether charge takes the form

$$G_{b_{i0}} \equiv K^i = \frac{mc(x^i - t\dot{x}^i)}{\sqrt{1 - \frac{\dot{\boldsymbol{x}}^2}{c^2} + \Lambda\left[\frac{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})^2 - \boldsymbol{x}^2\dot{\boldsymbol{x}}^2}{3c^2} + \frac{(\boldsymbol{x}-\dot{\boldsymbol{x}}t)^2}{3}\right]}} = mc(x^i - t\dot{x}^i)\Gamma . \tag{77}$$

Considering $\ddot{\boldsymbol{x}} = 0$ and $\dot{\Gamma} = 0$, we can easily checked that K^i is really conserved in the dS/AdS-SR mechanics,

$$\dot{K}^i = 0, \quad \text{or} \quad \dot{\boldsymbol{K}} = 0 . \tag{78}$$

4. Angular momentum:

   Finally, we derive the angular momentum of dS/AdS-SR mechanics. Taking $b_{12} = -1$, other $b_{\mu\nu} = 0$ and $a_0 = a_1 = a_2 = a_3 = 0$ in the Killing vector, we obtain

$$t' = t, \quad x'^1 = x^1 - \epsilon x^2, \quad x'^2 = x^2 + \epsilon x^1, \quad x'^3 = x^3, \tag{79}$$

   and

$$\zeta = 0, \quad \eta^1 = -x^2, \quad \eta^2 = x^1, \quad \eta^3 = 0 . \tag{80}$$

   The conserved quantity is determined to be

$$G_{b_{12}} \equiv L^3 = \frac{m(x^1\dot{x}^2 - x^2\dot{x}^1)}{\sqrt{1 - \frac{\dot{\boldsymbol{x}}^2}{c^2} + \Lambda\left[\frac{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})^2 - \boldsymbol{x}^2\dot{\boldsymbol{x}}^2}{3c^2} + \frac{(\boldsymbol{x}-\dot{\boldsymbol{x}}t)^2}{3}\right]}} . \tag{81}$$

   For the choices of $b_{23} = -1$ and $b_{13} = -1$, the resulting Noether charges $G_{b_{23}} \equiv L^1 =$ and $G_{b_{13}} \equiv -L^2$ can be calculated as follows

$$L^i = \frac{m\epsilon^{ijk}x^j\dot{x}^k}{\sqrt{1 - \frac{\dot{\boldsymbol{x}}^2}{c^2} + \Lambda\left[\frac{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})^2 - \boldsymbol{x}^2\dot{\boldsymbol{x}}^2}{3c^2} + \frac{(\boldsymbol{x}-\dot{\boldsymbol{x}}t)^2}{3}\right]}} = m\epsilon^{ijk}x^j\dot{x}^k\Gamma , \tag{82}$$

   where ϵ^{ijk} is the totally antisymmetric Levi-Civita symbol. It is easy to verify that the angular momentum conservation law holds in dS/AdS-SR mechanics:

$$\dot{L}^i = 0, \quad \text{or} \quad \dot{\boldsymbol{L}} = 0 . \tag{83}$$

So far all the ten independent conserved Killing-Noether charges $\{E, \boldsymbol{p}, \boldsymbol{K}, \boldsymbol{L}\}$ have been found out. Comparing them with the corresponding results in [16] it is found that the Killing-Noether charges and the Noether charges deduced from the classical domain are exactly the same. Since the essential correctness of the classical domain method is less known in the community, our calculations in above are meaningful and useful for trusting in that method.

Existence of ten independent conserved Noether charges indicates also that the 4-dimension Beltrami spacetime has the maximal symmetry, and metric $B_{\mu\nu}(x)$ satisfies the 3$^{\text{rd}}$ requirement of basic metric for SR.

5 Summary and discussion

We show in this paper that when non-zero ECC (Einstein Cosmological Constant) Λ emerges as a universal parameter in the Einstein equation, the Minkowski spacetime metric $\eta_{\mu\nu}$ of the common Special Relativity is no longer a solution to the vacuum Einstein equation. This is a challenging puzzle in the relativity theories. The basic features for $\eta_{\mu\nu}$ are as follows: (i) It is the solution of vacuum Einstein equation with $\Lambda = 0$; (ii) The inertial motion law of a free particle holds true in the Minkowski spacetime, hence we call $\eta_{\mu\nu}$ inertial metric; (iii) It has maximal spacetime symmetry. In order to understand the puzzle mentioned above, we start from Ref. [2]. In Ref. [2], another inertial metric $g_{\mu\nu}(\lambda, x)$ with a parameter λ were found by a miracle, and it was called *the classical domain metric* originally in [2]. In the present paper, we have pursued this metric from two sides as follows:

1. Firstly, we successfully proved that when $\lambda = \Lambda/3$, the classical domain metric $g_{\mu\nu}(\lambda, x)$ satisfies the vacuum Einstein equation with $\Lambda \neq 0$, and named it Beltrami metric, i.e., $B_{\mu\nu}(x) = g_{\mu\nu}(\lambda, x)|_{(\lambda=\Lambda/3)}$. Thus, $B_{\mu\nu}(x)$ could be qualified to be the basic metrics of the dS/AdS-SR, if $B_{\mu\nu}(x)$ had maximal spacetime symmetry. Discussing the spacetime symmetry and the relevant physics is the main motivation of this work.

2. Secondly, therefore, we pay great attention to the Killing equations for $B_{\mu\nu}(x)$ and their general solutions. From Killing vector theory we gave the corresponding explicit expressions of the Killing equations which are system of ten simultaneous partial differential equations of first order. The general solutions for these Killing equations were obtained, and all ten independent Killing vectors were revealed explicitly. Such results confirm that the Beltrami metric $B_{\mu\nu}(x)$ has maximal spacetime symmetry. Since Killing vectors are the generators of the transformations preserving metric, the ten Killing-Noether charges should exist. The explicit form of these Noether charges have been calculated out. The results are just the energy, momenta, Lorentz boost and angular momentum $\{E, \boldsymbol{p}, \boldsymbol{K}, \boldsymbol{L}\}$ in SR-theory with $\Lambda \neq 0$.

The pioneer work on dS/AdS-SR [2] was based on the unpopular classical domain method. The present paper reformulates the theory of dS/AdS-SR by means of Killing vector geometric theory. Moreover, the study about the effects of vacuum with non-zero Einstein Cosmologic Constant is essential.

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